Where does Bernoulli's Equation come from?

Introduction

By now, you have seen the following equation many times, using it to solve simple fluid problems.

$$\frac{P}{\rho} + \frac{v^2}{2} + gz = constant\ (along\ a\ streamline)$$

This is known as *Bernoulli's Equation*. Let's explore this equation more deeply to better understands it's applications and limitations.

Before we get started, it's helpful to discuss the basic ways to visualize and solve fluid flow problems.

Streamlines

These are the most common way to visualize fluid patterns. Very convenient to calculate mathematically.

A line everywhere tangent to the velocity vector at a given instant.

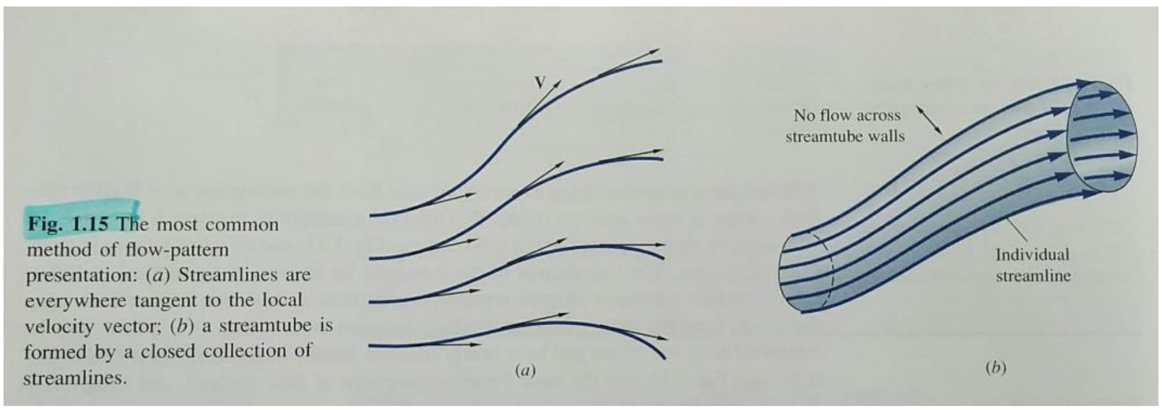


Fig. 1.15 The most common method of flow-pattern presentation: (*a*) Streamlines are everywhere tangent to the local velocity vector; (*b*) a streamtube is formed by a closed collection of streamlines.

Basic Flow Analysis Techniques

There are 3 basic ways to solve fluid flow problems:

1. Control volume or integral analysis
   - Work with a finite region making a balance of flows in and out to determine gross flow effects such as the force or torque on a body or the total energy exchange
   - Large scale
   - Convenient way to find the rate of change of an arbitrary gross fluid property
   - Often based on average property values at the boundaries
   - Useful for engineering "estimates"
2. Infinitesimal or differential analysis
   - Describe the detailed flow pattern at every point
   - Small scale

- Applies to any problem but differential equations almost always require computational analysis
- Limited opportunity for analytical solutions

3. Experimental study

In all cases, the flow must satisfy the following basic laws:

1. Conservation of mass (continuity)
2. Linear momentum
3. Conservation of energy (1st law of thermodynamics)
4. A state relation like $\rho = \rho(p, T)$
5. Appropriate boundary conditions at solid surfaces, interfaces, inlets and exits

Systems versus Control Volumes

System - arbitrary quantity of mass of fixed identity

Surroundings - Everything external to the system

Boundaries - Separates the system from it's surroundings

Laws of Mechanics - State what happens when there is an interaction between the system and it's surroundings.

1. Conservation of Mass (continuity)
   a. The mass of the system is conserved and does not change
   b. $m_{sys} = const$
   c. $\frac{dm}{dt} = 0$
2. Linear Momentum Relation
   a. 2nd Law of Thermodynamics
   b. If the surroundings apply a net force, F, on the system, then that mass will begin to accelerate:

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv)$$

3. Angular Momentum Relation
   a. If the surroundings exert a net moment M about the center of mass of the system, there will be a rotational effect.
   b. Not covered in detail in this class
4. Energy Conservation
   a. If heat δQ is added to the system or work δW is done by the system, the system energy δE must change according to the energy balance (1st Law of Thermodynamics)

$$\delta Q - \delta W = dE, \text{ or}$$

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

Control Volumes - convert the system laws to apply to a specific region, which the system may only occupy for an instant. The system passes on and other systems come along. The basic laws are reformulated to apply to the local region called a control volume.

Volume and Mass Rate of Flow

Volume flow rate (Q)

- amount of volume of fluid that passes through a surface
- positive value = outflow
- negative value = inflow

Mass flow ($\dot{m}$)

- the amount of mass of fluid that passes through a surface
- $\dot{m} = \rho Q$

Reynolds Transport Theorem

To convert a system analysis to a control volume analysis, we must convert our mathematics to apply to a specific region, rather than to individual masses.

This conversion is called the Reynolds Transport Theorem.

We need to relate the time derivative of a system property to the rate of change of that property within a certain region.

Control volumes can either be fixed, moving, or deformable. The form of the transport theorem changes slighting depending on what type of CV you choose.

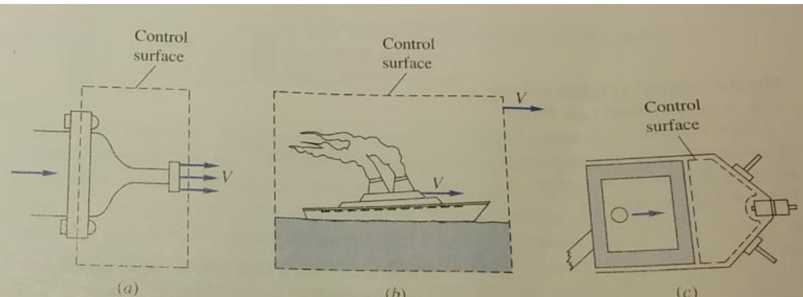


Fig. 3.2 Fixed, moving, and deformable control volumes: (*a*) fixed control volume for nozzle stress analysis; (*b*) control volume moving at ship speed for drag force analysis; (*c*) control volume deforming within cylinder for transient pressure variation analysis.

One dimensional fixed control volume - Simplified example

- Choose *B* to represent a property of the fluid (energy, momentum, volume flow)
- Let $\beta = \frac{dB}{dm}$, the amount of *B* per unit mass in any small portion of fluid

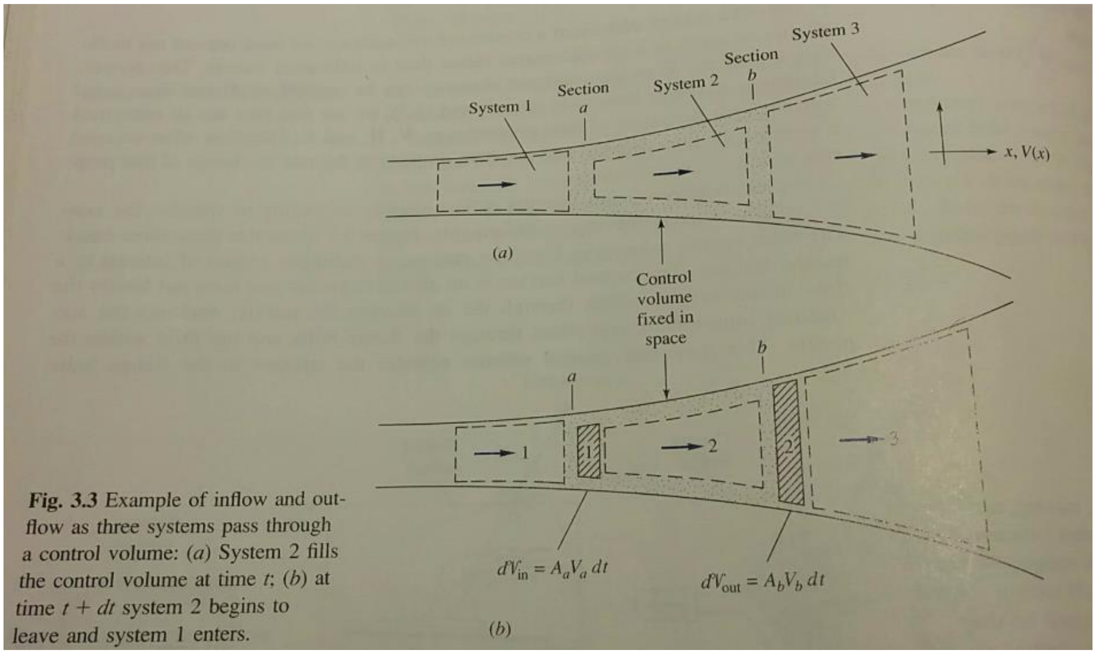


Fig. 3.3 Example of inflow and outflow as three systems pass through a control volume: (*a*) System 2 fills the control volume at time *t*; (*b*) at time *t* + *dt* system 2 begins to leave and system 1 enters.

- Reynolds Transport Theorem states that

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{cv} \beta\rho dV\right) + (\beta\rho AV)_{out} - (\beta\rho AV)_{in}$$

  - The three terms on the right side are:
    - The rate of change of *B* within the control volume = <u>0 for steady flow</u>
    - The flux of B passing out of the control surface
    - The flux of B passing in to the control surface

Bernoulli's Equation

Bernoulli's is an approximate relation between *pressure*, *velocity* and *elevation*, but is only applicable in regions of steady, incompressible flow where net friction forces are negligible, where:

- Steady state flow where velocity does not change with time at a specified location, and
- All particles that pass through the same point follow the same path (streamline), and
- Velocity vectors remain tangent to the path

Inviscid Fluid

The key approximation is that viscous effects are negligible (aka. "Inviscid fluid"). While this assumption cannot be valid for the entire flow field, it can be a reasonable approximation in certain regions of many practical flows. These regions are referred to as inviscid regions of flow and are characterized as regions where net viscous or frictional forces are negligible compared to other forces acting on the fluid including pressure and gravity forces.

Regions that are NOT inviscid regions include close to solid walls (boundary layers) and directly downstream of bodies (wakes).

Figure 5-21 shows where Bernoulli may be applied along a streamline and where it is not applicable.

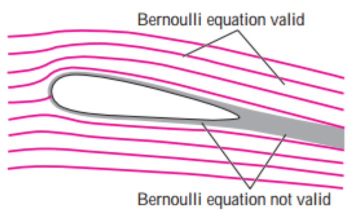
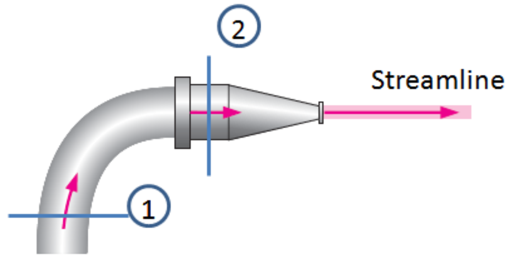


FIGURE 5–21
The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of boundary layers and wakes.

Bernoulli's Equation States that:

$$\frac{P}{\rho} + \frac{v^2}{2} + gz = constant\ (along\ a\ streamline)$$

Therefore, in the following streamline



$$\frac{P_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{v_2^2}{2} + gz_2$$

This formula states that pressure can convert to velocity and/or elevation head.

The fact that these units are the same is not entirely obvious, so let's show their equivalence using dimensional analysis.

- $\frac{P}{\rho} \rightarrow \frac{N/m^2}{kg/m^3} = \frac{N}{m^2} \times \frac{m^3}{kg} = \frac{kg \cdot m}{s^2} \times \frac{m}{kg} = \frac{m^2}{s^2}$ ODD UNITS.
- $\frac{v^2}{2} \rightarrow \frac{m^2}{s^2}$
- $gz \rightarrow \frac{m}{s^2} \times m = \frac{m^2}{s^2}$

$N = \frac{kg \cdot m}{s^2}$, $J = N \cdot m$

SPECIFIC ENERGY $= \frac{J}{kg} = \frac{N \cdot m}{kg} = \frac{\frac{kg \cdot m}{s^2} \times m}{kg} = \frac{m^2}{s^2}$

∴ UNITS OF BERNOULLI EQUATION

→ SPECIFIC ENERGY $= \frac{J}{kg} = \frac{m^2}{s^2}$

Acceleration of Fluid Particle

The velocity of a particle is related to the distance by:

$$v = \frac{ds}{dt}$$

Which may vary along a streamline. In 2D flow, the acceleration can be decomposed into components:

1. Streamwise acceleration (a_s)
    a. Change in speed along the streamline
2. Normal acceleration (a_n)
    a. Direction is normal to the streamline
    b. $a_n = \frac{v^2}{R}$
    c. Acceleration due to change in direction. For straight paths, $a_n = 0$.

Steady state implies that there is no change with time at a specified location, but the value of a quantity may change from one location to another.

$$dv = \frac{\partial v}{\partial s} ds + \frac{\partial v}{\partial t} dt$$

Meaning the change in velocity along a streamline is a defined by how it changes with position over an infinitely small distance as well as how it changes velocity with time, over an infinitely small length of time.

If we divide this equation by ***dt***, we get

$$\frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t}\frac{dt}{dt}$$

And

$$\frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t}$$

In steady flow, where $\dot{m}_1 = \dot{m}_2$, then $\frac{\partial v}{\partial t} = 0$, meaning that *v = v(s).* Thus:

$$a_s = \frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{ds}{dt} = \frac{\partial v}{\partial s} v = v\frac{dv}{ds}$$

Derivation of the Bernoulli Equation

Consider the motion of a fluid particle as depicted below.

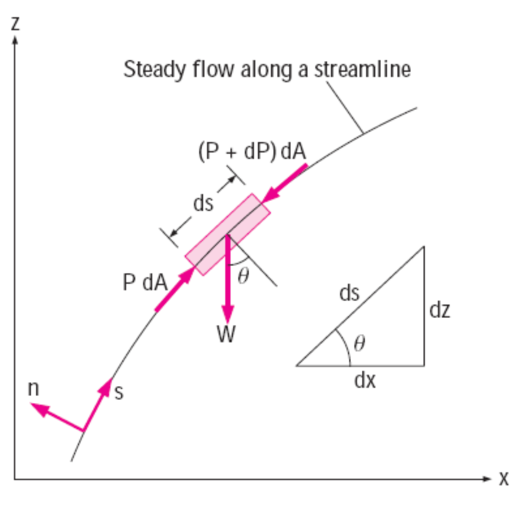


FIGURE 5–23
The forces acting on a fluid particle along a streamline.

Applying conservation of linear momentum (2nd law):

$$\sum F_s = ma_s$$

Assuming inviscid flow, friction forces are assumed to be negligible. The significant forces acting on the fluid particle include:

- Pressure acting on both sides
- The component of the particles weight in the s-direction

Therefore,

$$PdA - (P + dP)dA - Wsin\theta = mv\frac{dv}{ds}$$

We know that:

- The mass of the particle can be described as $m = \rho V = \rho dAds$
- The weight of the particle, $W = mg = \rho gdAds$
- $sin\theta = dz/ds$

Substituting these into our equation:

$$-dPdA - \rho gdA\, ds\frac{dz}{ds} = \rho dA\, ds\, v\frac{dv}{ds}$$

Divide by *dA*

$$-dP - \rho g\, dz = \rho\ \, v\, dv$$

Remembering some calculus, $v\, dv = \frac{1}{2}d(v^2)$ and dividing each term by ρ

$$\frac{dP}{\rho} + \frac{1}{2}d(v^2) + g\, dz = 0$$

Integrating this equation yields

$$\int\frac{dP}{\rho} + \frac{v^2}{2} + gz = constant$$

Assuming incompressible flow, $\int\frac{dP}{\rho} = \frac{P}{\rho}$

Therefore we have derived Bernoulli's equation for steady, incompressible flow along a streamline in inviscid regions of flow.

$$\frac{P}{\rho} + \frac{v^2}{2} + gz = constant$$

In other words: The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when the compressibility and frictional effects are negligible.