Buoyancy

The two laws of buoyancy, discovered by Archimedes in the 3rd century are:

1. A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces
2. A floating body displaces its own weight in the fluid in which it floats.

2.4 BUOYANCY

The body shown in Figure 2.9 is submerged in a fluid with density ρ. The resultant force **F** holds the body in equilibrium.

The element of volume $h\,dA$ has gravity and pressure forces acting on it. The component of the force due to the pressure on the top of the element is $-P_2\,dS_2\cos\alpha\,\mathbf{e}_y$, where α is the angle between the plane of the element dS_2 and the xz plane. The product $dS_2 \cos\alpha$ then is the projection of dS_2 onto the xz plane, or simply dA. The net

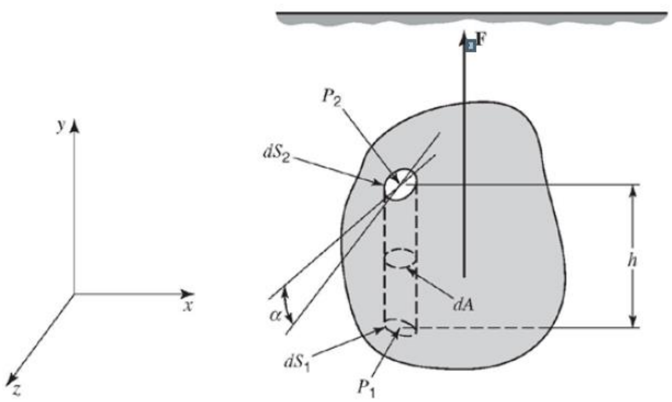


Figure 2.9 Forces on submerged volume.

pressure force on the element is $(P_1 - P_2)\,dA\,\mathbf{e}_y$, and the resultant force on the element is

$$d\mathbf{F} = (P_1 - P_2)\,dA\,\mathbf{e}_y - \rho_B g h\,dA\,\mathbf{e}_y$$

where ρ_B is the density of the body. The difference in pressure $P_1 - P_2$ may be expressed as $\rho g h$, so

$$d\mathbf{F} = (\rho - \rho_B) g h\,dA\,\mathbf{e}_y$$

Integration over the volume of the body, assuming constant densities, yields

$$\mathbf{F} = (\rho - \rho_B) g V \mathbf{e}_y \tag{2-8}$$

where V is the volume of the body. The resultant force **F** is composed of two parts, the weight $-\rho_B g V \mathbf{e}_y$ and the buoyant force $\rho g V \mathbf{e}_y$. The body experiences an upward force equal to the weight of the displaced fluid. This is the well-known principle of Archimedes. When $\rho > \rho_B$, the resultant force will cause the body to float on the surface. In the case of a floating body, the buoyant force is $\rho g V_s \mathbf{e}_y$, where V_s is the submerged volume.

Some more supporting documentation:

$$dA = dS_2 cos\alpha$$

$$dP = \frac{dF}{dA}$$

$$dF = dP \cdot dA = dP \cdot dS_2 cos\alpha$$

We know that:

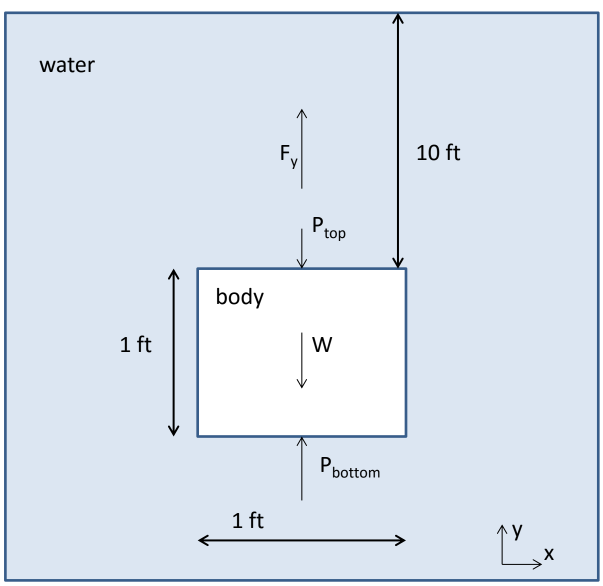
$$\frac{dP}{dy} e_y = -\rho g e_y$$

$$|P| = -\rho g = P_2$$

Therefore

$$F_2 = -P_2 \cdot dS_2 cos\alpha e_y$$

EXAMPLE 6	A cube measuring 1 ft on a side is submerged so that its top face is 10 ft below the free surface of water. Determine the magnitude and direction of the applied force necessary to hold the cube in this position if it is made of **(a)** cork ($\rho = 10\,\text{lb}_m/\text{ft}^3$) **(b)** steel ($\rho = 490\,\text{lb}_m/\text{ft}^3$).



Remember that F = P/A, therefore P = FA. We know that A = 1ft x 1 ft = 1ft^2

The pressure forces on all lateral surfaces of the cube cancel. Those on the top and bottom do not, as they are at different depths.

Summing forces on the vertical direction, we obtain

$$\Sigma F_y = -W + P(1)|_{\text{bottom}} - P(1)|_{\text{top}} + F_y = 0$$

where F_y is the additional force required to hold the cube in position.

Expressing each of the pressures as $P_{\text{atm}} + \rho_w g h$, and W as $\rho_c g V$, we obtain, for our force balance

$$-\rho_c g V + \rho_w g(11\,\text{ft})(1\,\text{ft}^2) - \rho_w g(10\,\text{ft})(1\,\text{ft}^2) + F_y = 0$$

Where did P_{atm} go?

$$P_{top} = P_{atm} + \rho_w g h$$

$$P_{bottom} = P_{atm} + \rho_w g(h+1)$$

$$P_{bottom} - P_{top} = P_{atm} + \rho_w g(h+1) - (P_{atm} + \rho_w g h)$$

$$P_{bottom} - P_{top} = P_{atm} + \rho_w g(11) - \rho_w g(10)$$

Solving for F_y, we have

$$F_y = -\rho_w g[(11)(1) - 10(1)] + \rho_c g V = -\rho_w g V + \rho_c g V$$

The first term is seen to be a buoyant force, equal to the weight of displaced water.

Finally, solving for F_y, we obtain

(a) $\rho_c = 10\ \text{lb}_\text{m}/\text{ft}^3$

$$F_y = -\frac{(62.4\,\text{lb}_\text{m}/\text{ft}^3)(32.2\,\text{ft/s}^2)(1\,\text{ft}^3)}{32.2\,\text{lb}_\text{m}\text{ft/s}^2\,\text{lb}_\text{f}} + \frac{(10\,\text{lb}_\text{m}\text{ft}^3)(32.2\,\text{ft/s}^2)(1\,\text{ft}^3)}{32.2\,\text{lb}_\text{m}\text{ft/s}^2\,\text{lb}_\text{f}}$$

$$= -52.4\,\text{lb}_\text{f}\ (\text{downward})(-233\,\text{N})$$

(b) $\rho_c = 490\,\text{lb}_\text{m}/\text{ft}^3$

$$F_y = -\frac{(62.4\,\text{lb}_\text{m}/\text{ft}^3)(32.2\,\text{ft/s}^2)(1\,\text{ft}^3)}{32.2\,\text{lb}_\text{m}\text{ft/s}^2\,\text{lb}_\text{f}} + \frac{(490\,\text{lb}_\text{m}\text{ft}^3)(32.2\,\text{ft/s}^2)(1\,\text{ft}^3)}{32.2\,\text{lb}_\text{m}\text{ft/s}^2/\text{lb}_\text{f}}$$

$$= +427.6\,\text{lb}_\text{f}(\text{upward})(1902\,\text{N})$$

In case (a), the buoyant force exceeded the weight of the cube, thus to keep it submerged 10 ft below the surface, a downward force of over 52 lb was required. In the second case, the weight exceeded the buoyant force, and an upward force was required.