ME 305 Fluid Mechanics I

Part 7

Dimensional Analysis and Similitude

These presentations are prepared by

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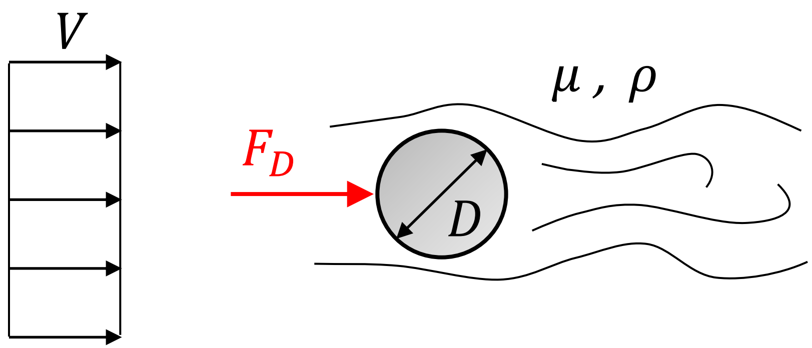
Dimensional Analysis

- Consider that we are interested in determining how the drag force acting on a smooth sphere immersed in a uniform flow depends on other fluid and flow variables.

- Important varibles of the problem are shown below (How did we decide on these?).



- Drag force F_D is thought to depend on the following variables.

$$F_D = f(D, V, \mu, \rho)$$

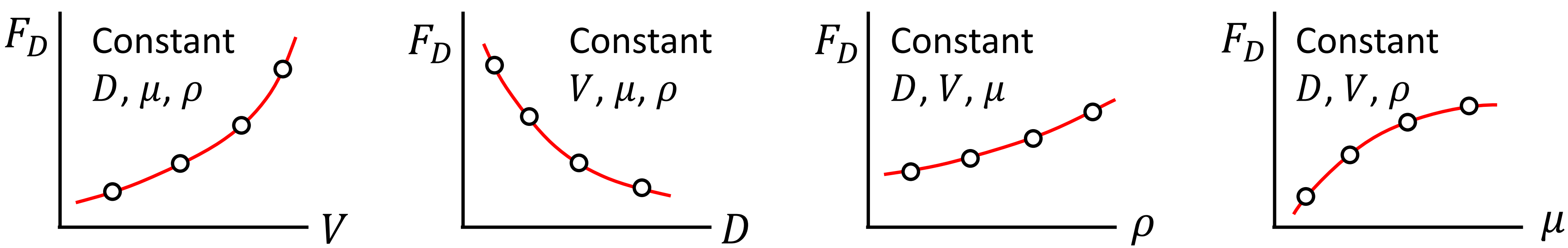
- In order to find the actual functional relation we need to perform a set of experiments.

- Dimensional analysis helps us to design and perform these experiments in a systematic way.

Dimensional Analysis (cont'd)

- The following set of controlled experiments should be done.
  - Fix D, μ and ρ. Change V and measure F_D.
  - Fix V, μ and ρ. Change D and measure F_D.
  - Fix D, V and μ . Change ρ and measure F_D.
  - Fix D, V and ρ. Change μ and measure F_D.

Note : These are only illustrative figures. They do not correspond to any actual experimentation.



- We need to perform too many experiments.
- Also there are major difficulties such as finding fluids with different densities, but same viscosities.

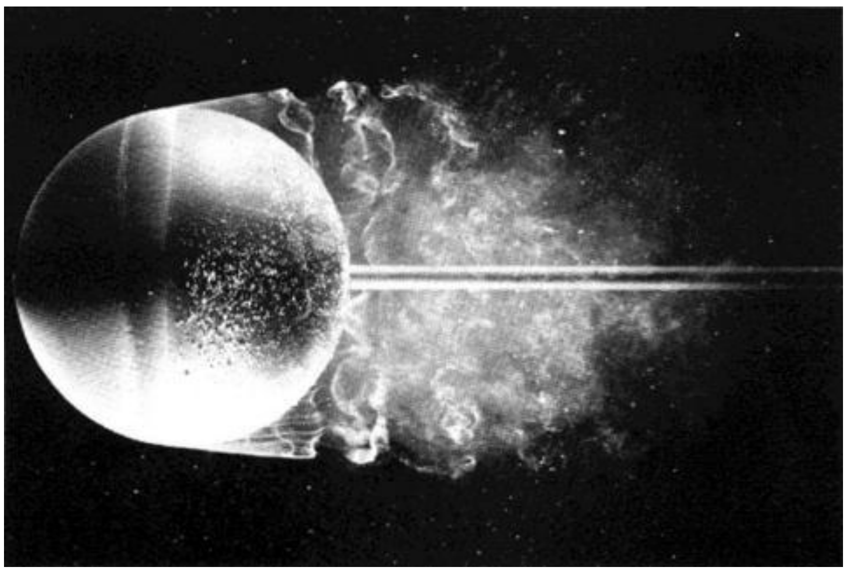
Dimensional Analysis (cont'd)

- It is possible to simplify the dependency of drag force on other variables by using nondimensional (unitless) parameters.

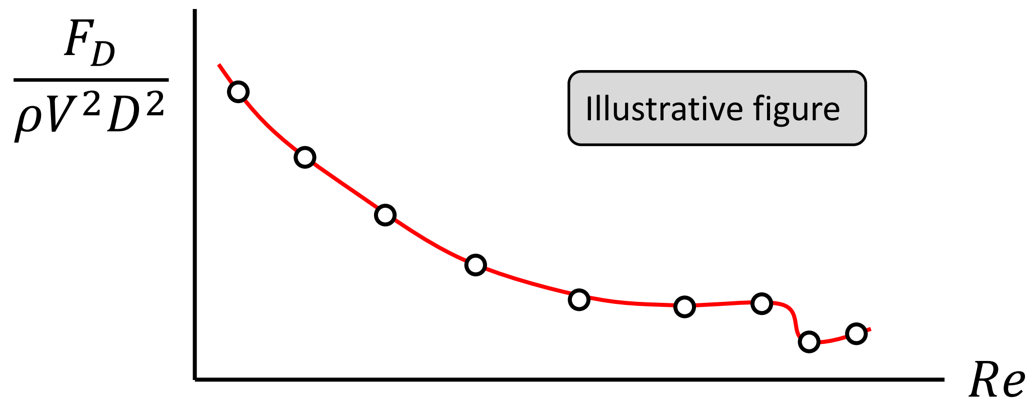
$$\frac{F_D}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

Nondimensional drag force

Nondimensional Reynolds number (Re)



Flow over a sphere at $Re = 15000$



Dimensional Analysis (cont'd)

- To find this new relation, we only need to change the Reynolds number.
- We can do it in any way we want, e.g. the simplest way is to change the speed of air flow in a wind tunnel.
- All $Re = 15000$ flows around a sphere will look like the same and they all provide the same nondimensional drag force. It does not matter what fluid we use or how big the sphere is (be aware of very extreme cases).
- Dimensional analysis is used to formulate a physical phenomenon as a relation between a set of nondimensional (unitless) groups of variables such that the number of these groups is less than the number of dimensional variables.
- It is important to develop a systematic and meaningful way to perform experiments.
- Nature of the experiments are simplified and the number of required experiments is reduced.

Buckingham Pi Theorem

- Buckingham Pi theorem can be used to determine the nondimensional groups of variables (Pi groups) for a given set of dimensional variables.
- For the flow over a sphere problem studied previously, dimensional parameter set is (F_D, D, V, μ, ρ) and this theorem helps us to find two Pi groups as

$$\Pi_1 = \frac{F_D}{\rho V^2 D^2} \qquad \text{and} \qquad \Pi_2 = \frac{\rho V D}{\mu}$$

- Let's explain how this works using "the drag force acting on a sphere" problem.
- Step 1 : List all the dimensional variables involved in the problem.
- n is the number of dimensional variables. $n = 5$ for our example.
- These variables should be independent of each other. For example if the diameter of a sphere is in the list, frontal area of the sphere can not be included.
- If body forces are important in a problem, gravitational acceleration should be in the list, although it is a constant.

Buckingham Pi Theorem (cont'd)

- Step 2 : Express each of the variables in tems of basic dimensions, which are

L : length , T : time , M : mass

- For problems involving heat transfer Θ (temperature) can also be a basic dimension.
- For the example we are studying basic dimensions of variables are

$$[F_D] = \left[\frac{ML}{T^2}\right] \quad , \quad [D] = [L] \quad , \quad [V] = \left[\frac{L}{T}\right] \quad , \quad [\rho] = \left[\frac{M}{L^3}\right] \quad , \quad [\mu] = \left[\frac{M}{LT}\right]$$

- Our example involves $r = 3$ primary dimensions. For most fluid mechanics problems r will be 3.

- Variables having only L in their dimension are called geometric variables.
- Variables having only T or both L and T are called kinematic variables.
- Variables having M in their dimension are called dynamic variables.
- For our example D is a geometric, V is a kinematic and F_D , μ and ρ are dynamic variables.

Buckingham Pi Theorem (cont'd)

- Step 3: Determine the repeating variables that are allowed to appear in more than one Pi group.
- There should be r many repeating variables.
- If L is a primary dimension of the problem, we should select one geometric variable as a repeating variable.
- If T is a primary dimension of the problem, we should select one kinematic variable as a repeating variable.
- If M is a primary dimension of the problem, we should select one dynamic variable as a repeating variable.
- Note that this selection is not unique and the resulting Pi groups will depend on our selection. Certain selections are ‘‘better’’ than others.
- For the problem of interest we can select D , V and ρ as repeating variables.
- If there is an obvious dependent variable in the problem, do not select it as a repeating variable. In our example F_D is a dependent variable. We are trying to understand how it depends on other variables.

Buckingham Pi Theorem (cont'd)

- Step 4: Determine $(n-r)$ many Pi groups by combining repeating variables with nonrepeating variables and using the fact that Pi groups should be nondimensional.
- For our example we need to find $5-3=2$ Pi groups. Each Pi group will include only one of the nonrepating variables.

$$\Pi_1 = F_D \underbrace{D^a \; V^b \; \rho^c}$$

A nonrepeating parameter

Unknown combination of repeating parameters

We need to determine a, b and c.

- Π_1 should be unitless : $[-] = \left[\frac{ML}{T^2}\right] [L]^a \left[\frac{L}{T}\right]^b \left[\frac{M}{L^3}\right]^c$

Π_1 should have no L dimension : $0 = 1 + a + b - 3c$

Π_1 should have no T dimension : $0 = -2 - b$

Π_1 should have no M dimension : $0 = 1 + c$

$$a = -2, \quad b = -2, \quad c = -1$$

$$\Pi_1 = \frac{F_D}{\rho D^2 V^2}$$

Buckingham Pi Theorem (cont'd)

- Now determine the second Pi group which has μ as the nonrepeating variable.

$$\Pi_2 = \mu \; D^a \; V^b \; \rho^c$$

- Π_2 should be unitless : $[-] = \left[\frac{M}{LT}\right] [L]^a \left[\frac{L}{T}\right]^b \left[\frac{M}{L^3}\right]^c$

$$\left.\begin{array}{ll} \Pi_2 \text{ should have no } L \text{ dimension} : & 0 = -1 + a + b - 3c \\ \Pi_2 \text{ should have no } T \text{ dimension} : & 0 = -1 - b \\ \Pi_2 \text{ should have no } M \text{ dimension} : & 0 = 1 + c \end{array}\right\} \left.\begin{array}{l} a = -1 \\ b = -1 \\ c = -1 \end{array}\right\} \Pi_2 = \frac{\mu}{\rho D V}$$

- Therefore the relation of nondimensional groups that we are after is

$$\Pi_1 = f_1(\Pi_2) \qquad \rightarrow \qquad \frac{F_D}{\rho V^2 D^2} = f_1\left(\frac{\mu}{\rho V D}\right)$$

- It is better to write the second Pi group as $\frac{\rho V D}{\mu}$ because it is the well known Reynolds number.

Exercises for Buckingham Pi Theorem

Exercise : Consider the flow of an incompressible fluid through a long, smooth-walled horizontal, circular pipe. We are interested in analyzing the pressure drop, Δp, over a pipe length of L. Other variables of the problem are pipe diameter (D), average velocity (V) and fluid properties (ρ and μ). Determine the Pi groups by a) selecting ρ as a repeating parameter, b) selecting μ as a repeating parameter.

Exercise : In a laboratory experiment a tank is drained through an orifice from initial liquid level h_0. The time, τ, to drain the tank depends on tank diameter, D, orifice diameter, d, gravitational acceleration, g, liquid properties, ρ and μ. Determine the Pi groups.

Exercise : The diameter, d, of the dots made by an ink jet printer depends on the ink properties, ρ and μ, surface tension, σ, nozzle diameter, D, the distance, L, of the nozzle from the paper and the ink jet velocity, V. Determine the Pi groups.

Exercise : The power, $\mathcal{P}$, required to drive a propeller is known to depend on the following variables: freestream speed, V, propeller diameter, D, angular speed, ω, fluid properties, ρ and μ, and the speed of sound c. Determine the Pi groups.

Important Nondimensional Numbers of Fluid Mechanics

- Following nondimensional numbers frequently appear as a Pi group.
- Reynolds number : $Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$. Ratio of inertia forces to viscous forces.
- Euler number : $Eu = \frac{\Delta p}{\frac{1}{2}\rho V^2}$. Ratio of pressure forces to inertia forces.
- Froude number : $Fr = \frac{V}{\sqrt{gL}}$. Squareroot of the ratio of inertia forces to gravitational forces.
- Mach number : $Ma = \frac{V}{\sqrt{E_v/\rho}} = \frac{V}{c}$. Squareroot of the ratio of inertia forces to compressibility forces.
- Weber number : $We = \sqrt{\frac{\rho V^2 L}{\sigma}}$. Squareroot of the ratio of inertia forces to surface tension forces.
- Strouhal number : $St = \frac{\omega L}{V}$. Used for flows with oscillatory (periodic) behavior.
- Cavitation number : $Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$. Used for possibly cavitating flows.

Model and Prototype

- In experimental fluid mechanics we sometimes can not work with real sized objects, known as prototypes.
- Instead we use scaled down (or up) versions of them, called models.
- Also sometimes in experiments we use fluids that are different than actual working fluids, e.g. we use regular tap water instead of salty sea water to test the performance of a marine propeller.

Wind tunnel tests of an airliner

Race car being tested in a water tunnel

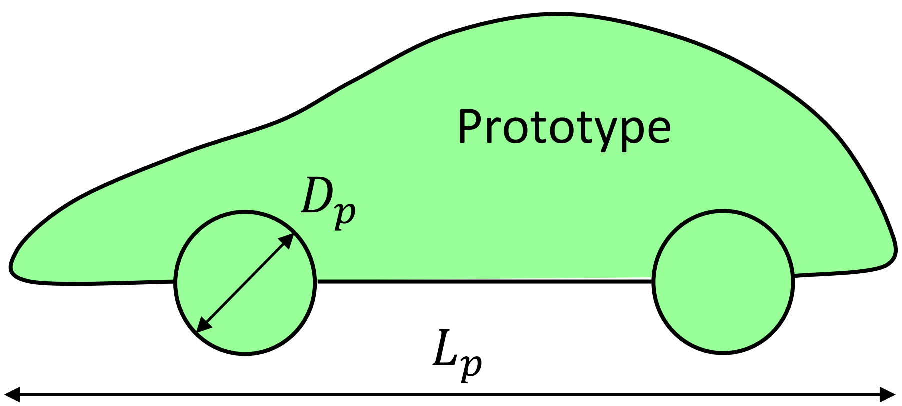
http://www.reuters.com/news/video/story?videoId=131255095

Three Basic Laws of Similitude

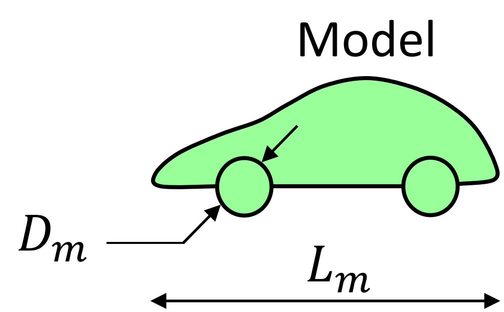
- A similitude analysis is done to make sure that the results obtained from an experiment can correctly be transferred to the real flow field.
- Three basic laws of similitude must be satisfied in order to achieve complete similarity between prototype and model flow fields.

1. Geometric similarity : Model and prototype must be the same in shape, but can be different in size. All linear dimensions of the model be related to corresponding dimensions of the prototype by a constant length ratio, L_r.

    - It is usually impossible to establish 100 % geometric similarity due to very small details that can not be put into the model. Modeling surface roughness exactly is also impossible.



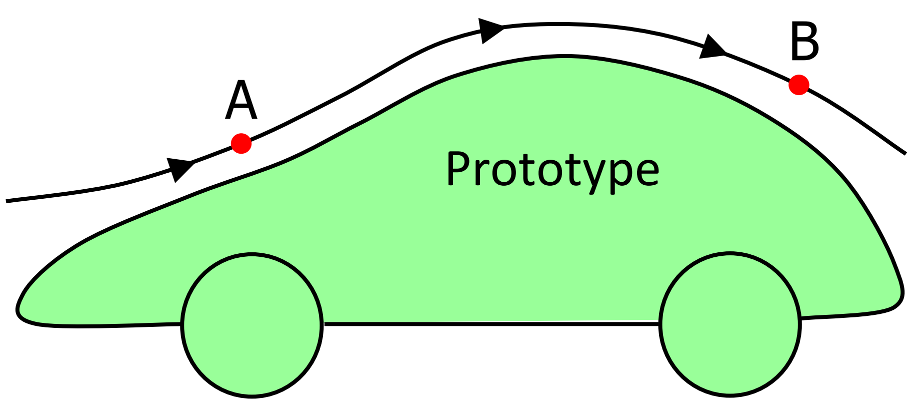
$$L_r = \frac{L_p}{L_m} = \frac{D_p}{D_m}$$



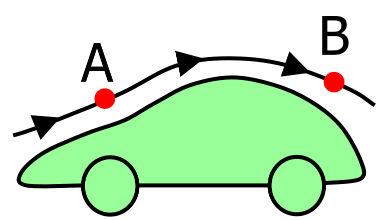
Three Basic Laws of Similitude (cont'd)

2. Kinematic similarity : Model and prototype flow fields are kinematically similar if the velocities at corresponding points are the same in direction and differ only by a constant factor of velocity ratio, V_r.

- This also means that the streamline patterns of two flow fields should differ by a constant scale factor.



$$V_r = \frac{V_{p_A}}{V_{m_A}} = \frac{V_{p_B}}{V_{m_B}}$$



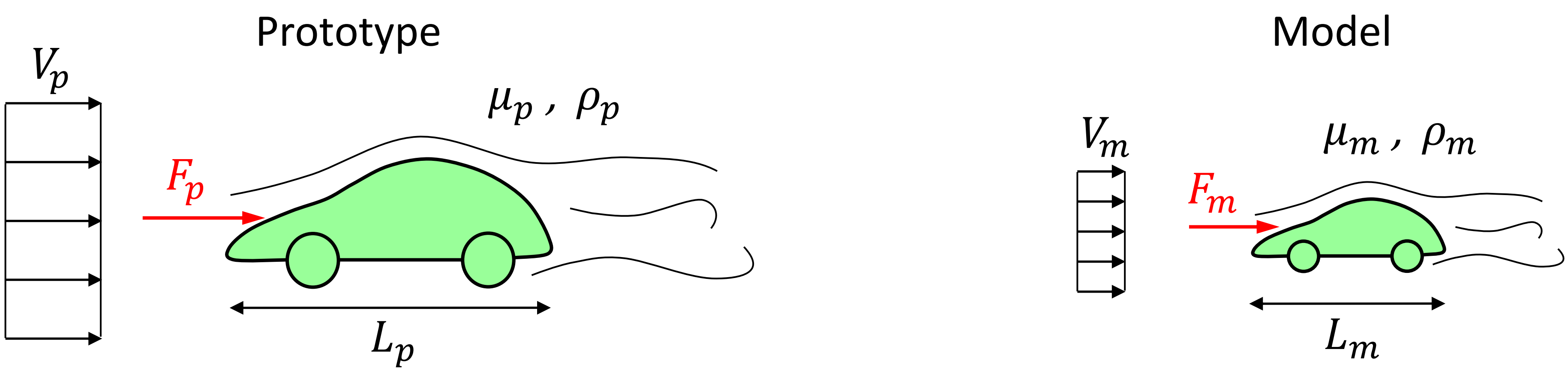
Three Basic Laws of Similitude (cont'd)

3. Dynamic similarity : Two flow fields should have force distributions such that identical types of forces are parallel and are related in magnitude by a constant factor of force ratio.

- If a certain type of force, e.g. compressibility force, is highly dominant in the prototype flow, it should also be dominant in the model flow.
    - How suitable would it be to use a water tunnel to study the aerodynamic forces acting on a supersonic missile ?
- If a certain type of force, e.g. surface tension force, is negligibly small in the prototype flow, it should also be small in the model flow.
    - How suitable should it be to use a very light and very small model to test the forces acting on a ship ?
- To establish dynamic similarity we need to determine the important forces of the prototype flow and make sure that the nondimensional numbers related to those forces are the same in prototype and model flows.

Similitude (cont'd)

Exercise : The aerodynamic drag of a new car is to be predicted at a speed of 100 km/h at an air temperature of 25 °C. Engineers build a one fifth scale of the car to test in a wind tunnel. It is winter and the air in the tunnel is only about 5 °C. Determine how fast the engineers should run the wind tunnel in order to achieve similarity between the model and the prototype. How each Newton of drag force measured on the model be transferred to the prototype ?



$$Re_p = Re_m$$

$$\frac{\rho_p V_p D_p}{\mu_p} = \frac{\rho_m V_m D_m}{\mu_m}$$

Similitude (cont'd)

- The important question is how to decide on the important force types for a given problem? In other words equality of which nondimensional numbers should be sought?
- Reynolds number similarity is important for almost all flows.
- Froude number similarity is important for flows with free surfaces, such as ship resistance, open channel flows and for flows driven by the action of gravity.
- Euler number similarity is important mostly for turbomachinery flows with considerable pressure changes, for which cavitation may be a concern.
- Mach number similarity is important for high speed flows.
- Weber number similarity is important for problems involving interfaces between two fluids and low weight objects.
- Strouhal number similarity is important for flows with an oscillating (time periodic) flow pattern, such as von Karman vortices shed from bodies.

Similitude Exercises

Exercise : The drag force on a submarine, which is moving well below the free surface, is to be determined by a test on a model, which is scaled down to one-twentieth of the prototype. The test is to be carried in a water tunnel. The density and kinematic viscosity of the seawater are 1010 kg/m^3 and 1.3×10^{-6} m^2/s. The water in the tunnel has a density of 988 kg/m^3 and a kinematic viscosity of 0.65×10^{-6} m^2/s. If the speed of the prototype is 2.6 m/s, then determine the

a) speed of the model.

b) ratio of the drag force in the prototype to the one in the model.

Similitude Exercises (cont'd)

Exercise : The model described in the previous problem will now be used to determine the drag force of a submarine, which is moving on the surface. The properties of the sea water are as given above. The speed of the prototype is 2.6 m/s.

a) Determine the speed of the model.

b) Determine the kinematic viscosity of the liquid that should be used in the experiments.

c) If such a liquid is not available, sea will be used in the experiments. the viscous affects, determine the drag force due to the This is called incomplete

Similitude Exercises (cont'd)

Exercise : A model of a harbor is to be made with a scale ratio of 300. Storm waves having amplitude of 2 m occur on the breakwater of prototype harbor at a speed of 10 m/s.

a) Neglecting the frictional effects, determine the amplitude and speed of the waves in the model.

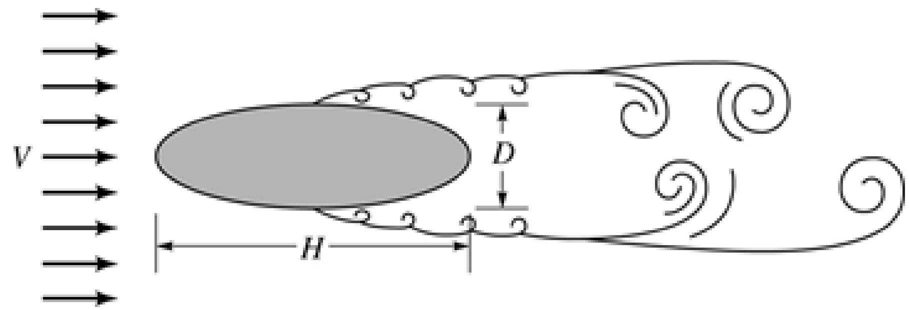
b) If the tidal period in the prototype is 12 h, determine the tidal period in the model.

Exercise : An airplane travels in air at a velocity of 200 m/s. Pressure and temperature of air are 90 kPa and 10ºC, respectively. A model of this airplane with a length scale of 10 is tested in a wind tunnel at 20ºC. The specific heat ratio and gas constant for air are 1.4 and 287.1 J/kgK, respectively. Taking the effect of compressibility into account, determine the

a) velocity of air in the wind tunnel,

b) density of air in the wind tunnel.

Similitude Exercises (cont’d)



Exercise : A long structural component of a bridge has an elliptical cross section. It is known that when an unsteady wind blows past this type of bluff body, vortices may develop on the downwind side that are shed in a regular fashion at some definite frequency. Since these vortices can create harmful periodic forces acting on the structure, it is important to determine the shedding frequency. For the specific structure of interest, $D = 0.1$ m, $H = 0.3$ m, and a representative wind velocity is 50 km/hr. Standard air can be assumed. The shedding frequency is to be determined through the use of a small scale model that is to be tested in a water tunnel. For the model, $D_m = 20$ mm and the water temperature is 20 °C.

Determine the model dimension, H_m, and the velocity at which the test should be performed. If the shedding frequency for the model is found to be 49.9 Hz, what is the corresponding frequency for the prototype?

Movie
Collapse of Tacoma Narrows Bridge
http://www.youtube.com/watch?v=j-zczJXSxnw