Meng 263 – Fluids and Heat Transfer

Time Transient Heat Transfer – Lump Parameter Model

Something that is hot tends to cool.

Engineers want to know how quickly something is cooling and how long it will take for something to cool to a certain temperature. Engineering is all about prediction.

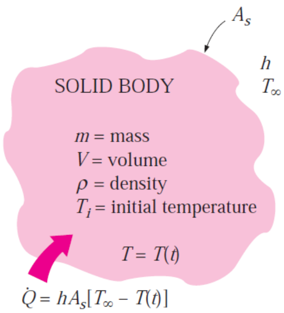


FIGURE 18–2
The geometry and parameters involved in the lumped system analysis.

Consider a body of arbitrary shape of mass m, volume V, surface area A_s, density ρ, and specific heat C_p initially at a uniform temperature T_i (Fig. 18–2). At time $t = 0$, the body is placed into a medium at temperature T_∞, and heat transfer takes place between the body and its environment, with a heat transfer coefficient h. For the sake of discussion, we will assume that $T_\infty > T_i$, but the analysis is equally valid for the opposite case. We assume lumped system analysis to be applicable, so that the temperature remains uniform within the body at all times and changes with time only, $T = T(t)$.

During a differential time interval dt, the temperature of the body rises by a differential amount dT. An energy balance of the solid for the time interval dt can be expressed as

$$\begin{pmatrix}\text{Heat transfer into the body}\\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the}\\ \text{energy of the body}\\ \text{during } dt\end{pmatrix}$$

or

$$hA_s(T_\infty - T)\,dt = mC_p\,dT \qquad \textbf{(18–1)}$$

Noting that $m = \rho V$ and $dT = d(T - T_\infty)$ since T_∞ = constant, Eq. 18–1 can be rearranged as

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho VC_p}\,dt \qquad \textbf{(18–2)}$$

Integrating from $t = 0$, at which $T = T_i$, to any time t, at which $T = T(t)$, gives

$$\ln\frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho VC_p}\,t \qquad \textbf{(18–3)}$$

Taking the exponential of both sides and rearranging, we obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \tag{18–4}$$

where

$$b = \frac{hA_s}{\rho V C_p} \quad \text{(1/s)} \tag{18–5}$$

is a positive quantity whose dimension is $(\text{time})^{-1}$. The reciprocal of b has time unit (usually s), and is called the **time constant.** Equation 18–4 is plotted in Fig. 18–3 for different values of b. There are two observations that can be made from this figure and the relation above:

1. Equation 18–4 enables us to determine the temperature $T(t)$ of a body at time t, or alternatively, the time t required for the temperature to reach a specified value $T(t)$.
2. The temperature of a body approaches the ambient temperature T_∞ exponentially. The temperature of the body changes rapidly at the beginning, but rather slowly later on. A large value of b indicates that the body will approach the environment temperature in a short time. The larger the value of the exponent b, the higher the rate of decay in temperature. Note that b is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body. This is not surprising since it takes longer to heat or cool a larger mass, especially when it has a large specific heat.

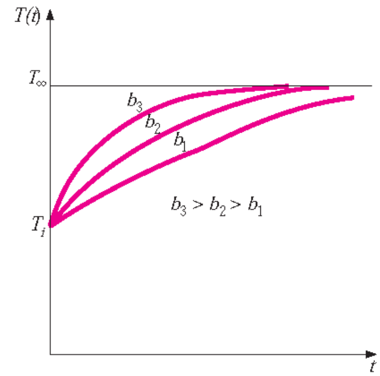


FIGURE 18–3
The temperature of a lumped system approaches the environment temperature as time gets larger.

Once the temperature $T(t)$ at time t is available from Eq. 18–4, the *rate* of convection heat transfer between the body and its environment at that time can be determined from Newton's law of cooling as

$$\dot{Q}(t) = hA_s[T(t) - T_\infty] \quad \text{(W)} \tag{18–6}$$

The *total amount* of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t is simply the change in the energy content of the body:

$$Q = mC_p[T(t) - T_i] \quad \text{(kJ)} \tag{18–7}$$

The amount of heat transfer reaches its *upper limit* when the body reaches the surrounding temperature T_∞. Therefore, the *maximum* heat transfer between the body and its surroundings is (Fig. 18–4)

$$Q_{max} = mC_p(T_\infty - T_i) \quad \text{(kJ)} \tag{18–8}$$

We could also obtain this equation by substituting the $T(t)$ relation from Eq. 18–4 into the $\dot{Q}(t)$ relation in Eq. 18–6 and integrating it from $t = 0$ to $t \to \infty$.

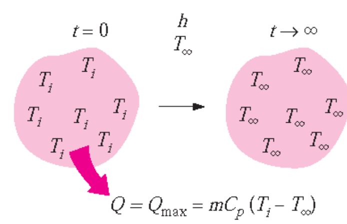


FIGURE 18–4
Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.

Criteria for Lumped System Analysis

The lumped system analysis certainly provides great convenience in heat transfer analysis, and naturally we would like to know when it is appropriate to use it. The first step in establishing a criterion for the applicability of the lumped system analysis is to define a **characteristic length** as

$$L_c = \frac{V}{A_s}$$

and a **Biot number** Bi as

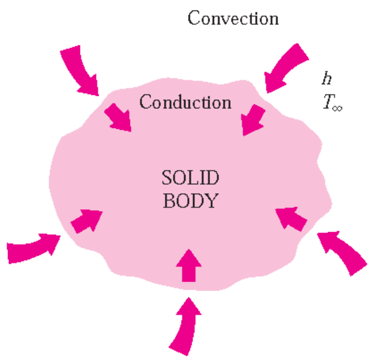
$$\text{Bi} = \frac{hL_c}{k} \tag{18–9}$$

It can also be expressed as (Fig. 18–5)

$$\text{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

or

$$\text{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$



$$\text{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

FIGURE 18–5
The Biot number can be viewed as the ratio of the convection at the surface to conduction within the body.

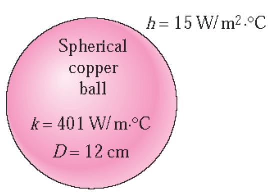
When a solid body is being heated by the hotter fluid surrounding it (such as a potato being baked in an oven), heat is first *convected* to the body and subsequently *conducted* within the body. The Biot number is the *ratio* of the internal resistance of a body to *heat conduction* to its external resistance to *heat convection*. Therefore, a small Biot number represents small resistance to heat conduction, and thus small temperature gradients within the body.

Lumped system analysis assumes a *uniform* temperature distribution throughout the body, which will be the case only when the thermal resistance of the body to heat conduction (the *conduction resistance*) is zero. Thus, lumped system analysis is *exact* when $\text{Bi} = 0$ and *approximate* when $\text{Bi} > 0$. Of course, the smaller the Bi number, the more accurate the lumped system analysis. Then the question we must answer is, How much accuracy are we willing to sacrifice for the convenience of the lumped system analysis?

Before answering this question, we should mention that a 20 percent uncertainty in the convection heat transfer coefficient h in most cases is considered "normal" and "expected." Assuming h to be *constant* and *uniform* is also an approximation of questionable validity, especially for irregular geometries. Therefore, in the absence of sufficient experimental data for the specific geometry under consideration, we cannot claim our results to be better than ± 20 percent, even when $\text{Bi} = 0$. This being the case, introducing another source of uncertainty in the problem will hardly have any effect on the overall uncertainty, provided that it is minor. It is generally accepted that lumped system analysis is *applicable* if

$$\text{Bi} \le 0.1$$

When this criterion is satisfied, the temperatures within the body relative to the surroundings (i.e., $T - T_\infty$) remain within 5 percent of each other even for well-rounded geometries such as a spherical ball. Thus, when $\text{Bi} < 0.1$, the variation of temperature with location within the body will be slight and can reasonably be approximated as being uniform.



$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02 \text{ m}$$

$$\text{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

FIGURE 18–6
Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.

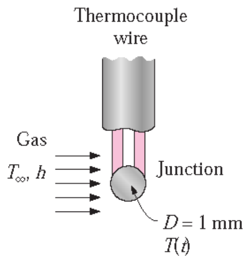


FIGURE 18–9
Schematic for Example 18–1.

EXAMPLE 18–1 Temperature Measurement by Thermocouples

The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1-mm-diameter sphere, as shown in Fig. 18–9. The properties of the junction are $k = 35$ W/m · °C, $\rho = 8500$ kg/m^3, and $C_p = 320$ J/kg · °C, and the convection heat transfer coefficient between the junction and the gas is $h = 210$ W/m^2 · °C. Determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference.

SOLUTION The temperature of a gas stream is to be measured by a thermocouple. The time it takes to register 99 percent of the initial ΔT is to be determined.

Assumptions **1** The junction is spherical in shape with a diameter of $D = 0.001$ m. **2** The thermal properties of the junction and the heat transfer coefficient are constant. **3** Radiation effects are negligible.

Properties The properties of the junction are given in the problem statement.

Analysis The characteristic length of the junction is

$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = \frac{1}{6}(0.001 \text{ m}) = 1.67 \times 10^{-4} \text{ m}$$

Then the Biot number becomes

$$\text{Bi} = \frac{hL_c}{k} = \frac{(210 \text{ W/m}^2 \cdot °\text{C})(1.67 \times 10^{-4} \text{ m})}{35 \text{ W/m} \cdot °\text{C}} = 0.001 < 0.1$$

Therefore, lumped system analysis is applicable, and the error involved in this approximation is negligible.

In order to read 99 percent of the initial temperature difference $T_i - T_\infty$ between the junction and the gas, we must have

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = 0.01$$

For example, when $T_i = 0°$C and $T_\infty = 100°$C, a thermocouple is considered to have read 99 percent of this applied temperature difference when its reading indicates $T(t) = 99°$C.

The value of the exponent b is

$$b = \frac{hA_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} = \frac{210 \text{ W/m}^2 \cdot °\text{C}}{(8500 \text{ kg/m}^3)(320 \text{ J/kg} \cdot °\text{C})(1.67 \times 10^{-4} \text{ m})} = 0.462 \text{ s}^{-1}$$

We now substitute these values into Eq. 18–4 and obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \quad \longrightarrow \quad 0.01 = e^{-(0.462 \text{ s}^{-1})t}$$

which yields

$$t = \mathbf{10 \text{ s}}$$

Therefore, we must wait at least 10 s for the temperature of the thermocouple junction to approach within 1 percent of the initial junction-gas temperature difference.

Discussion Note that conduction through the wires and radiation exchange with the surrounding surfaces will affect the result, and should be considered in a more refined analysis.

EXAMPLE 18–2 Predicting the Time of Death

A person is found dead at 5 PM in a room whose temperature is 20°C. The temperature of the body is measured to be 25°C when found, and the heat transfer coefficient is estimated to be $h = 8\ \text{W/m}^2 \cdot {}^\circ\text{C}$. Modeling the body as a 30-cm-diameter, 1.70-m-long cylinder, estimate the time of death of that person (Fig. 18–10).

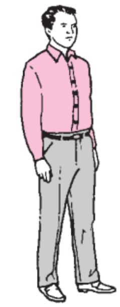


FIGURE 18–10
Schematic for Example 18–2.

SOLUTION A body is found while still warm. The time of death is to be estimated.

Assumptions **1** The body can be modeled as a 30-cm-diameter, 1.70-m-long cylinder. **2** The thermal properties of the body and the heat transfer coefficient are constant. **3** The radiation effects are negligible. **4** The person was healthy(!) when he or she died with a body temperature of 37°C.

Properties The average human body is 72 percent water by mass, and thus we can assume the body to have the properties of water at the average temperature of $(37 + 25)/2 = 31$°C; $k = 0.617$ W/m · °C, $\rho = 996$ kg/m^3, and $C_p = 4178$ J/kg · °C (Table A-15).

Analysis The characteristic length of the body is

$$L_c = \frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_o L + 2\pi r_o^2} = \frac{\pi(0.15\text{ m})^2(1.7\text{ m})}{2\pi(0.15\text{ m})(1.7\text{ m}) + 2\pi(0.15\text{ m})^2} = 0.0689\text{ m}$$

Then the Biot number becomes

$$\text{Bi} = \frac{hL_c}{k} = \frac{(8\text{ W/m}^2 \cdot {}^\circ\text{C})(0.0689\text{ m})}{0.617\text{ W/m} \cdot {}^\circ\text{C}} = 0.89 > 0.1$$

Therefore, lumped system analysis is *not* applicable. However, we can still use it to get a "rough" estimate of the time of death. The exponent b in this case is

$$b = \frac{hA_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} = \frac{8\text{ W/m}^2 \cdot {}^\circ\text{C}}{(996\text{ kg/m}^3)(4178\text{ J/kg} \cdot {}^\circ\text{C})(0.0689\text{ m})}$$
$$= 2.79 \times 10^{-5}\text{ s}^{-1}$$

We now substitute these values into Eq. 18–4,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{25 - 20}{37 - 20} = e^{-(2.79 \times 10^{-5}\text{ s}^{-1})t}$$

which yields

$$t = 43{,}860\text{ s} = \mathbf{12.2\ h}$$

Therefore, as a rough estimate, the person died about 12 h before the body was found, and thus the time of death is 5 AM. This example demonstrates how to obtain "ball park" values using a simple analysis.